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REPRINTED PAPER * by M. Caputo and F. Mainardi, from the journal *"Pure and Applied Geophysics (PAGEOPH)"*, vol. 91, No 1 (1971), pp. 134-147

EDITORIAL NOTE: Prof. Caputo is a Honorary Member of Editorial Board of our journal. This year 2007, together with his *80th personal anniversary*, we pay attention to *the 40th anniversary of the so-called "Caputo derivative"*. This variant of the operator for differentiation of fractional order, as oriented towards applications in mechanics – especially to linear models of viscoelasticity, appeared first in Caputo's paper of 1967, *"Linear Models of Dissipation whose Q is almost Frequency Independent – II"*, *Geophys. J. R. Astr. Soc.*, vol. **13**, No 5, Nov. 1967, pp. 529-539. The so called "PAGEOPH"-paper of 1971 by M. Caputo and F. Mainardi (then as his Ph.D. student), which we reproduce in the next 14 pages, continues the use of the Caputo derivative (operator (A.16), p. 146). However, the main contribution of this paper is of being *one of the first proposing a plot of the Mittag-Leffler function*, now recognised as the *Qeen-function of Fractional Calculus*. On those times (1971), without possibilities to take profit of software packages like *MATHEMATICA*, *MAPLE* and *MATLAB*, this task was difficult one as managing with series expansions convergent only in mathematical sense but not in the numerical sense. Some other authors worked also on similar problems either simultaneously but independently, or years afterwards. Although they recognised the Caputo - Mainardi achievements and priorities in private discussions, correspondence and public lectures, the Caputo-Mainardi PAGEOPH-1971 paper was often forgotten to be (or not properly) referred to . . . Therefore, we find it interesting and useful for our audience, as a journal closely specialized on Applied Fractional Calculus and Special Functions, to reproduce the paper in this jubillee issue.

* *Special thanks* are due, for the permission of 13 Nov. 2007 to reprint this article, to: *Birkhäuser Verlag AG (Basel)*, as Publisher of the journal *"Pure and Applied Geophysics"*. Currently, the soft copy of this paper (DOI: 10.1007/BF00879562) can be ordered via the SpringerLink Web-site, at: www.springerlink.com/content/wv5233j83145/?sortorder=asc&p_o=10

ON BEHALF OF THE EDITORIAL BOARD OF "FCAA",

Managing Editor: Virginia Kiryakova

A New Dissipation Model Based on Memory Mechanism

By M. CAPUTO and F. MAINARDI [1])

Summary – The model of dissipation based on memory introduced by Caputo is generalized and checked with experimental dissipation curves of various materials.

List of symbols

σ	unidimensional stress
ε	unidimensional strain
Q^{-1}	specific dissipation function
$c(t)$	creep compliance
$m(t)$	relaxation modulus
c_0	instantaneous compliance
m_∞	equilibrium modulus
$\psi(t)$	creep function
$\bar{\psi}(t)$	relaxation function
$\xi(\tau)$	spectral distribution of retardation times
$\bar{\xi}(\tau)$	spectral distribution of relaxation times
$c^*(\omega)$	complex compliance
$m^*(\omega)$	complex modulus
$\operatorname{tang}\delta$	loss-tangent

1. Introduction

The dissipation in solids may be produced by several different mechanisms, and although ultimately these all result in the mechanical energy being transformed into heat, two main dissipative processes are involved (KOLSKY [1] [2])).

The first type of process is known as 'static hysteresis'; the effect is that the energy loss per cycle (and hence the Q^{-1}) is independent on frequency. The principal cause may be associated simply with the 'static' non-linear stress-strain behaviour of the materials.

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[2]) Numbers in brackets refer to References, page 147.

The other type is attributed to 'viscosity', according to which many materials show losses which are associated with the velocity gradients set up by the vibrations. The forces producing these losses may be considered to be of a viscous nature and imply that the mechanical behaviour will depend upon the rate of straining. This is the subject of the 'linear viscoelasticity'.

It is possible to distinguish two types of viscous loss in solids and these correspond qualitatively to the behaviour of the two commonly employed rheological models, the Maxwell's and Kelvin-Voigt's.

But these models are too simple to conform phenomenologically to the behaviour of metals and rocks. The search for a proper model for such materials, which would account for their viscoelastic nature, has been the subject of many studies.

ZENER [2] for the metals, and MEIDAV [3] for the rocks, have shown that a linear combination of the Maxwell and Kelvin-Voigt models, known as the 'standard linear solid', becomes closer to giving an adequate description of the behaviour of such materials than either one of the two models.

CAPUTO [4, 5], has generalized the Kelvin-Voigt model by introducing a particular memory mechanism obtaining a good agreement with some experimental dissipation curves. This mechanism mathematically implies a convolution between the first order derivative of the strain and a kernel which is a power of the time; this operator is readily expressed by means of a derivative of real order.

In this paper we generalize the standard linear solid with the same type of memory mechanism as CAPUTO, and obtain a wider agreement with the experimental data.

2. *The stress-strain relations and the material functions*

As it is well known, the stress-strain relation for the standard linear solid (S.L.S.) may be written as follows

$$\left[1+\frac{1}{\beta}\frac{\partial}{\partial t}\right]\sigma(t)=m\left[1+\frac{1}{\alpha}\frac{\partial}{\partial t}\right]\varepsilon(t) \tag{1}$$

with

$$0<\alpha<\beta$$

where α, β are respectively the reciprocal of the retardation and relaxation times, and m is a suitable elastic modulus.

When $\alpha=\beta$, (1) simplifies to the relation valid for purely elastic material

$$\sigma=m\,\varepsilon$$

provided that $\sigma=\varepsilon=0$ when $t=0$.

When $(\beta-\alpha)/\alpha\ll 1$ the behaviour of the material is described as 'nearly elastic' (BERRY-HUNTER [6]).

Furthermore the current model contains the Maxwell and the Kelvin-Voigt models as limiting cases.

It seems to us that the action of the rates of stress and strain is not only instantaneous but still alive at later times. According to this idea, we introduce Caputo's memory mechanism into both stress and strain.

Following CAPUTO, we substitute into (1) the operator

$$\int_0^1 w(\mu)\frac{\partial^\mu}{\partial t^\mu}\,d\mu \tag{2}$$

instead of $\partial/\partial t$, where $\partial^\mu/\partial t^\mu$ is the derivative of order μ (see Appendix A) and $w(\mu)$ is a given 'weighting' function.

For sake of simplicity, we take

$$w(\mu) = \delta(\mu - \nu)$$

with $0 < \nu \leqq 1$, so that we write formally

$$\left[1 + \frac{1}{\beta}\frac{\partial^\nu}{\partial t^\nu}\right]\sigma(t) = m\left[1 + \frac{1}{\alpha}\frac{\partial^\nu}{\partial t^\nu}\right]\varepsilon(t). \tag{3}$$

When $\nu = 1$ we find of course (1); namely a three parameter model (S.L.S.); when $\nu \neq 1$ we obtain a new model depending on four parameters and hence exhibiting continuous spectra of retardation and relaxation times. This will be shown hereafter.

The response of the model to unit steps of stress and strain (the so-called material functions) is readily determined from (3), making use of the Laplace transform.

Following the notations introduced in the symbol list, we may express the material functions as follows:

$$c(t) = c_0 + \psi(t) \tag{4a}$$

$$m(t) = m_\infty + \bar{\psi}(t). \tag{4b}$$

Their Laplace transforms are respectively

$$C(p) = \frac{1}{p}\,\frac{1 + p^\nu/\beta}{m[1 + p^\nu/\alpha]} \tag{5a}$$

$$M(p) = \frac{1}{p}\,\frac{m[1 + p^\nu/\alpha]}{1 + p^\nu/\beta} \tag{5b}$$

where we have used (A.17).

Because of the initial and final values theorems, we obtain:

$$c_0 = \alpha/(m\,\beta) \tag{6a}$$

$$m_\infty = m. \tag{6b}$$

After easy calculations, the transforms of the creep and relaxation functions result:

$$\Psi(p) = \chi\left[\frac{1}{p} - \frac{p^{\nu-1}}{p^\nu + \alpha}\right] \tag{7a}$$

$$\bar{\Psi}(p) = \bar{\chi}\,\frac{p^{\nu-1}}{p^\nu + \beta} \tag{7b}$$

where

$$\chi \equiv \psi(\infty) = \frac{1}{m}\left(1 - \frac{\alpha}{\beta}\right) \tag{8a}$$

$$\bar{\chi} \equiv \bar{\psi}(0) = m\left(\frac{\beta}{\alpha} - 1\right). \tag{8b}$$

Carrying out the inversion of (7) and putting for convenience:

$$\alpha = (1/r)^{\nu} \tag{9a}$$

$$\beta = (1/\bar{r})^{\nu} \tag{9b}$$

we obtain:

$$\psi(t) = \chi\{1 - E_{\nu}[-(t/r)^{\nu}]\} \tag{10a}$$

$$\bar{\psi}(t) = \bar{\chi}\, E_{\nu}[-(t/\bar{r})^{\nu}] \tag{10b}$$

where E_{ν} denoting the Mittag-Leffler function of order ν (Erdelyi [7]).

This function was already proposed by Gross [8] in 1947 as an empirical law, in the attempt to eliminate the faults which a power law shows for the creep function. We are now deriving a similar result by introducing the memory mechanism into the stress strain relation.

A plot of the function $E_{\nu}[-\tau^{*\nu}]$ for some values of ν is shown in Fig. 1, where a decimal logarithmic scale has been used.

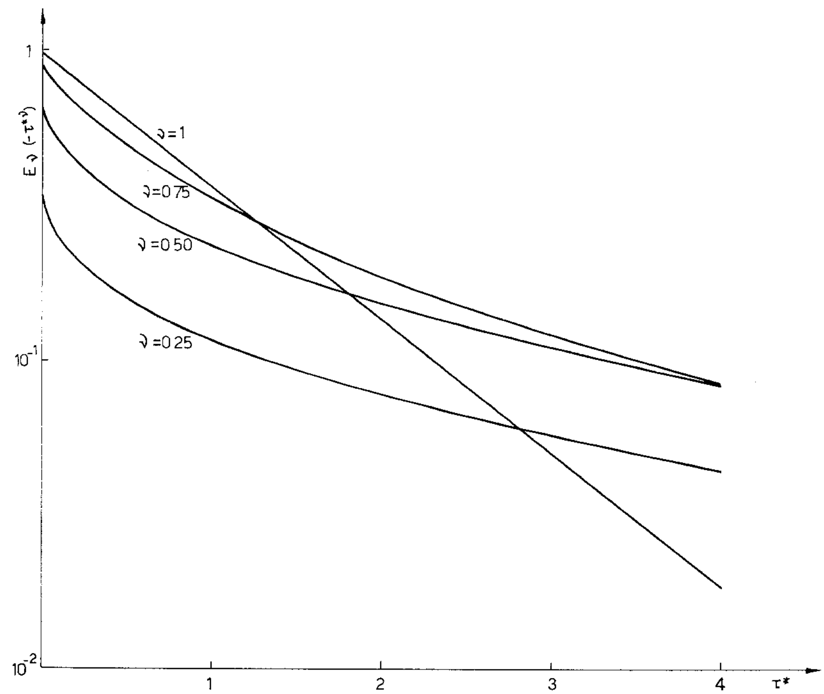

Figure 1

Mittag-Leffler function of order ν and argument $z = -\tau^{*\nu}$ for some values of ν

We recognize that the creep and relaxation functions reduce for $\nu = 1$ to exponential functions, as it must be for the S.L.S.; the constants r, $\bar{r}$ are respectively the single retardation and relaxation times. For $\nu \neq 1$ we obtain continuous spectra of retardation and relaxation times as ψ, $\bar{\psi}$ may be expressed in terms of the distribution functions ξ, $\bar{\xi}$ according to:

$$\psi(t) = \chi \int_0^\infty \xi(\tau)\,[1 - e^{-t/\tau}]\, d\tau \tag{11a}$$

$$\bar{\psi}(t) = \bar{\chi} \int_0^\infty \bar{\xi}(\tau)\, e^{-t/\tau}\, d\tau. \tag{11b}$$

The two spectra were determined by GROSS by means of a general method; it results

$$\xi(\tau)\, d\tau = \frac{1}{\pi} \frac{\sin(\nu\pi)}{(\tau/r)^\nu + (\tau/r)^{-\nu} + 2\cos(\nu\pi)} \frac{d\tau}{\tau} \tag{12a}$$

$$\bar{\xi}(\tau)\, d\tau = \frac{1}{\pi} \frac{\sin(\nu\pi)}{(\tau/\bar{r})^\nu + (\tau/\bar{r})^{-\nu} + 2\cos(\nu\pi)} \frac{d\tau}{\tau}. \tag{12b}$$

Plots of (12), for some values of ν are reported in Fig. 2. From these plots we can easily recognize the effect of ν on the character of the distributions; for $\nu \to 1$ the distributions become sharper and sharper until for $\nu = 1$ they reduce to delta functions.

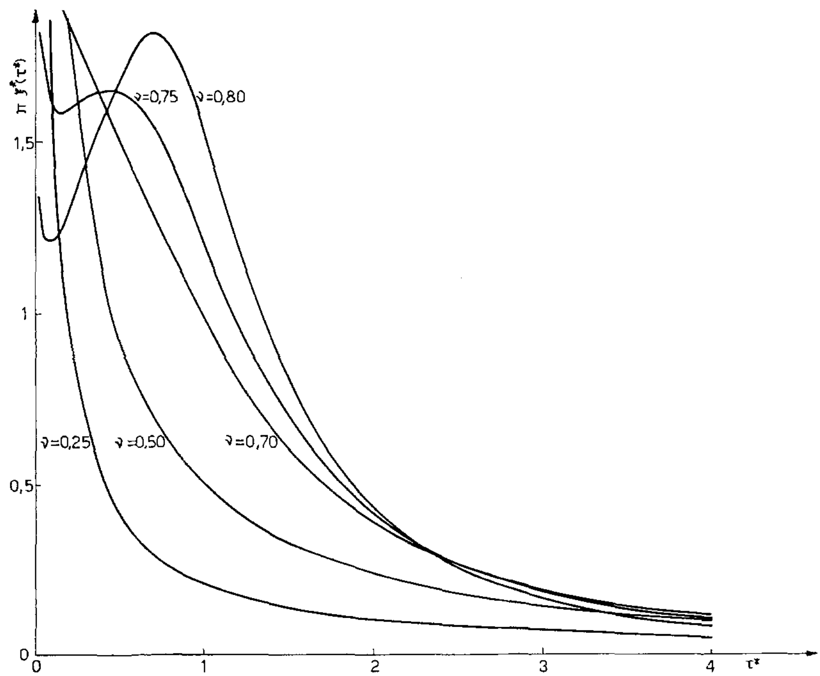

Figure 2
Spectral distributions of retardation/relaxation times for some values of ν

3. The specific dissipation function

The response of a linear viscoelastic body to sinusoidal excitations is determined by any of two complex functions, the complex compliance or the complex modulus, which depend on the driving frequency ω. These functions are related to the material functions according to

$$c^*(\omega) = p\, C(p)\big|_{p=i\omega} \tag{13a}$$

$$m^*(\omega) = p\, M(p)\big|_{p=i\omega}\,. \tag{13b}$$

The dissipation of the mechanical energy is related to the so-called loss-tangent

$$\operatorname{tang}\delta = \frac{-\operatorname{Im}[c^*(\omega)]}{\operatorname{Re}[c^*(\omega)]} = \frac{\operatorname{Im}[m^*(\omega)]}{\operatorname{Re}[m^*(\omega)]}. \tag{14}$$

For low-loss media (ZENER [2]) it is

$$\operatorname{tang}\delta \simeq Q^{-1}(\omega) \tag{15}$$

where Q^{-1} is the well-known specific dissipation function.

Making use of (5), (13), (14), (15), we may determine the Q^{-1} for our model; it is, after easy calculations

$$Q^{-1}(\omega) = (\beta - \alpha)\,\frac{\omega^{\nu}\sin(\nu\,\pi/2)}{\omega^{2\nu} + \alpha\,\beta + (\alpha+\beta)\,\omega^{\nu}\cos(\nu\,\pi/2)}. \tag{16}$$

In analogy with the S.L.S. we may refer the case $(\beta-\alpha)/\alpha \ll 1$ as to the nearly elastic one; this happens for the low-loss media.

Hereafter we give the approximative formula for Q^{-1} suitable to these media.

By putting

$$\delta = \beta - \alpha$$

$$\omega_0^{\nu} = \alpha$$

we obtain from (16), to the first order in δ

$$Q^{-1}(\omega) \simeq \delta\,\frac{\sin(\nu\,\pi/2)}{\omega^{2\nu} + \omega_0^{2\nu} + 2\,\omega_0^{\nu}\,\omega^{\nu}\cos(\nu\,\pi/2)}. \tag{17}$$

It is easy to recognize that ω_0 is the frequency at which (17) assumes its maximum given by

$$Q_{\mathrm{MAX}}^{-1} = \frac{\delta}{2\,\omega_0^{\nu}}\,\frac{\sin(\nu\,\pi/2)}{1+\cos(\nu\,\pi/2)}. \tag{18}$$

The (17), (18) will be used to check the validity of our model on account of the experimental data concerning some low-loss media.

4. *The experimental checks*

Experimental data on the specific dissipation are available for various viscoelastic solids; however measurements are always affected by considerable errors and, over large frequency range, are scarce because of considerable experimental difficulty.

For the metals indirect methods of measuring the Q^{-1} are used, i.e. the free oscillations and the resonance methods (KOLSKY [1], ZENER [2]). By these methods, BENNEWITZ and ROTGER [9, 10] measured the Q's for transverse vibrations in reeds of aluminium, brass, glass, German silver, silver, steel, in the frequency range between 10^0 and 10^3 cps. According to ZENER [2], thermal conduction provides the principal cause of dissipation in such experiments. Following this hypothesis, Zener demonstrated that the thermoelastic coupling changes the solid from a Hooke solid into an S.L.S., for which the Q^{-1} is given by (16) with $\nu=1$. The agreement between Zener's theory and Bennewitz-Rotger's experiments was remarkably good for German silver; for the other materials the fit was poor or impossible. We shall see that a satisfactory fit may be obtained with our model with $0<\nu\leq 1$ for all materials.

Experimental data on Q^{-1} are also available for the elastic waves in the earth's interior. CAPUTO [5] has reported the data on two curves: one related to the spheroidal oscillations and to the Rayleigh waves, the other to the tortional oscillations and to the Love waves. While a fit to the former curve has been possible by assuming Caputo's model (CAPUTO [4]), this does not occurs for the latter; we shall see that the present model provides in this case a satisfactory fit.

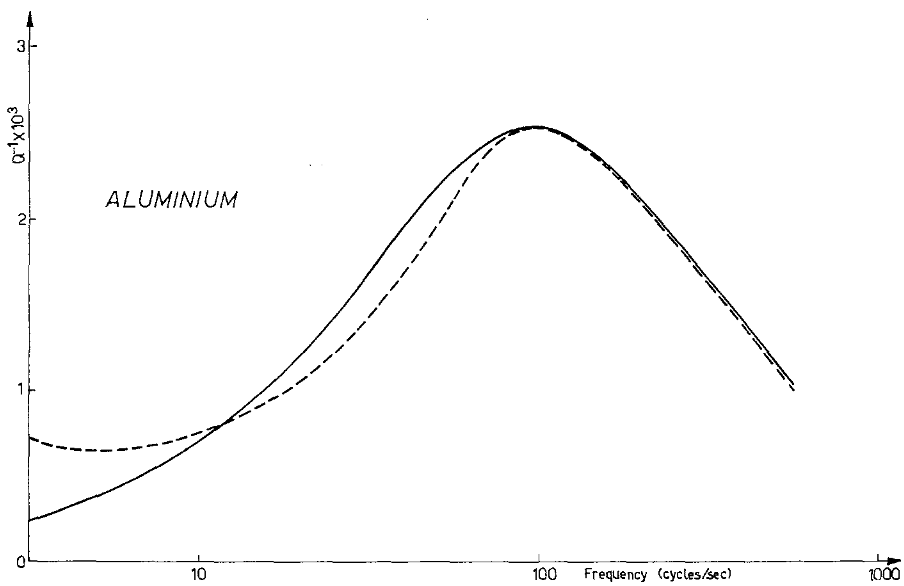

Figure 3
Specific dissipation in Aluminium: comparison between theoretical and experimental curves

In the attempt to obtaining the required fits, we find convenient to use the parameters δ, ω_0, ν of (17) as follows. From each data we find ω_0, Q_{MAX}^{-1}; then (18) is a relation between δ and ν. The theoretical curve, forced to pass through the maximum of the experimental curve, is then fitted to this by using the other free parameter.

The fits obtained for the metals are shown in Figs. (3) to (8), where a dashed line is used for the experimental curves and a continuous line for the theoretical ones.

For the earth the fit is shown in Fig. 9, where the data for torsional oscillations are reported on a dotted line and those for Love waves on a dashed line, while the continuous line shows as before the theoretical curve.

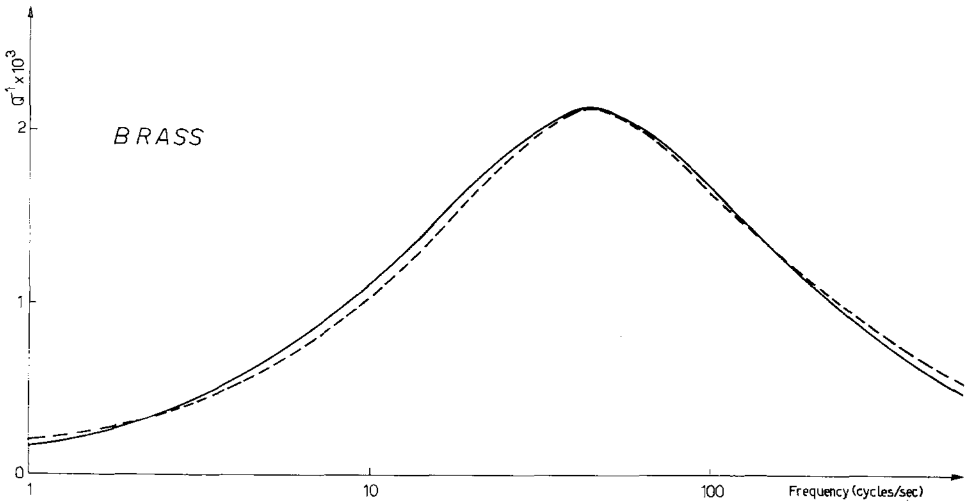

Figure 4
Specific dissipation in Brass: comparison between theoretical and experimental curves

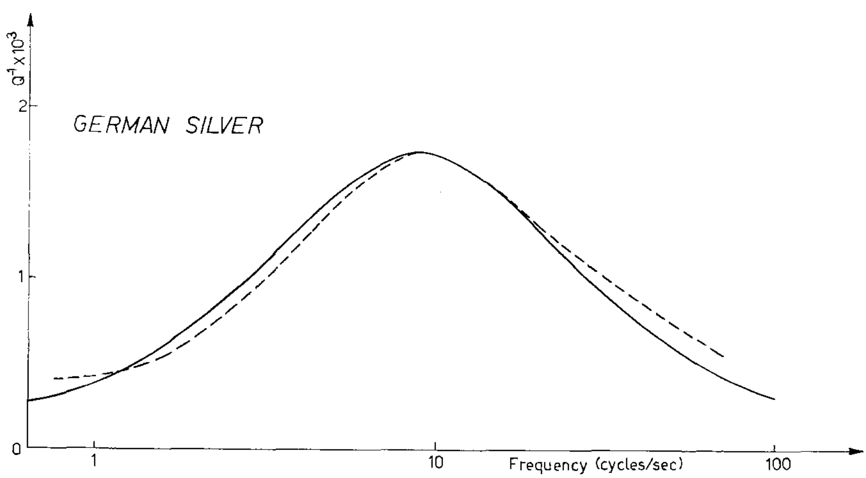

Figure 5
Specific dissipation in German silver: comparison between theoretical and experimental curves

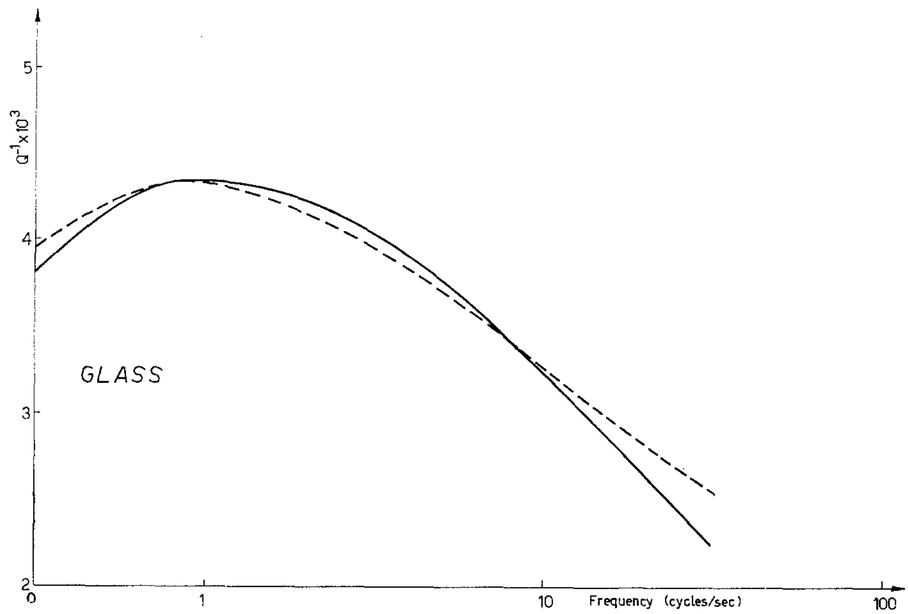

Figure 6
Specific dissipation in Glass: comparison between theoretical and experimental curves

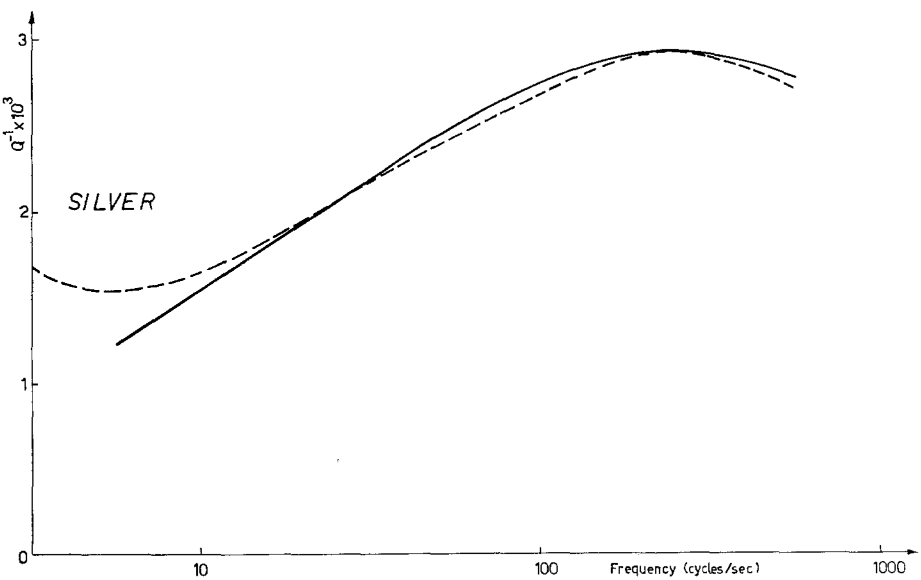

Figure 7
Specific dissipation in Silver: comparison between theoretical and experimental curves

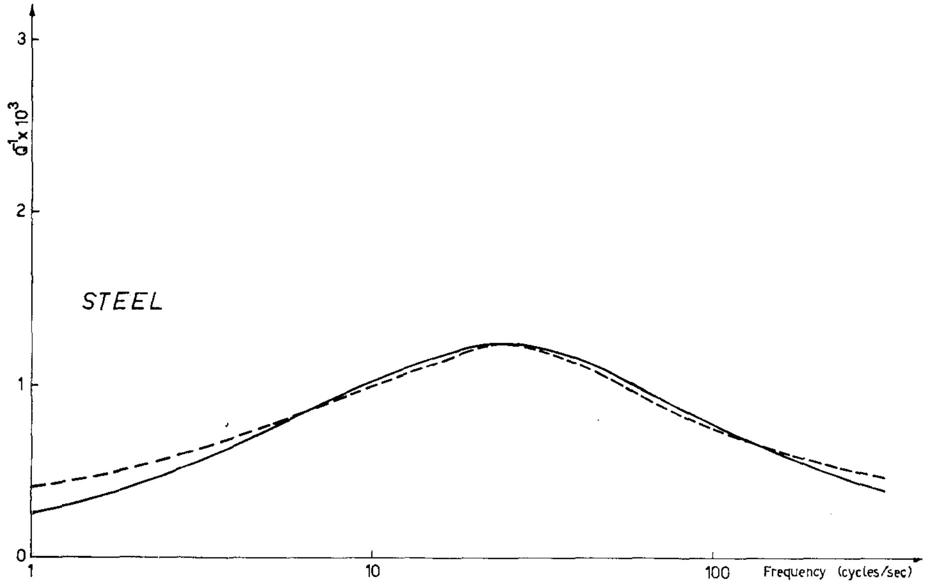

Figure 8
Specific dissipation in Steel: comparison between theoretical and experimental curves

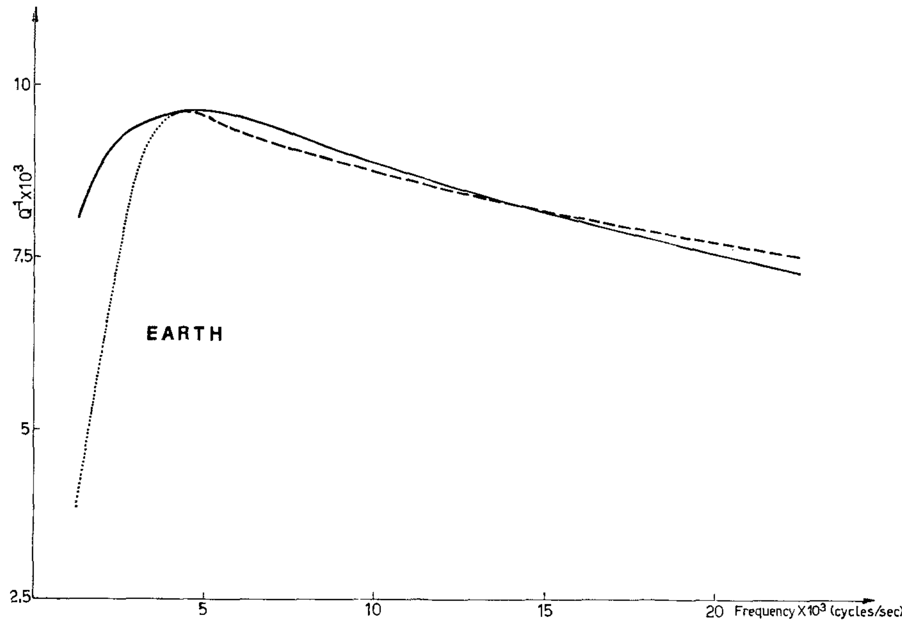

Figure 9
Specific dissipation in the Earth: comparison between theoretical and experimental curves

Table 1

Parameter values for the Q's fits

Parameters	δ (sec$^{-\nu}$)	α (sec$^{-\nu}$)	ν	$f_{max} = \omega_0/2\pi$ (cycles/sec)	Q^{-1}_{max}
Aluminium	1.88	316.5	0.90	95.5	2.54×10^{-3}
Brass	0.77	153.2	0.90	42.7	2.14×10^{-3}
German silver	0.19	56.0	1.00	8.9	1.74×10^{-3}
Glass	0.06	2.8	0.50	1.2	4.36×10^{-3}
Silver	0.55	39.3	0.50	245.5	2.93×10^{-3}
Steel	0.19	54.3	0.80	23.4	1.35×10^{-3}
Earth	4.37×10^{-3}	0.116	0.60	4.4×10^{-3}	9.58×10^{-3}

The parameter values of all the fits are listed in Table 1.

One can notice that for metals the agreement between the theory and the observed values is generally good. Say satisfactory for German silver, brass, glass, steel, and limited to a reduced extent in the frequency range around the maximum for aluminium and silver because their experimental curves show a minimum of Q^{-1}, which is not allowed by our model.

For the earth, the agreement is satisfactory except for very low frequencies; this discrepancy is expected and is in the right direction because of the low frequency waves sample deep material which is in different physical conditions.

Appendix A: The derivative of complex order

To introduce the notion of derivative of complex order we shall essentially follow GEL'FAND and SHILOV [11].

Cauchy's well known formula

$$\left.\begin{aligned} f_n(t) &= \int_0^t \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} \int_0^{\tau_1} f(\tau)\, d\tau\, d\tau_1 \ldots d\tau_{n-1} \\ &= \frac{1}{(n-1)!} \int_0^t f(\tau)\,(t-\tau)^{n-1}\, d\tau \quad (n = 1, 2, \ldots) \end{aligned}\right\} \tag{A.1}$$

reduces the calculation of the n-fold primitive of a function $f(t)$ defined for $t \geqq 0$ to a single integral. This formula may also be written in the from

$$f_n(t) = f(t) * \frac{t^{n-1}}{(n-1)!} \tag{A.2}$$

where for $t<0$ both $f(t)$ and t^{n-1} are replaced by zero and $*$ is denoting the convolution between ordinary functions.

It would seem quite natural to generalize this formula to the case of complex index λ, ($\operatorname{Re}\lambda>0$) and arbitrary generalized function f concentrated on the half-line $t\geqq 0$.

Then one defines the primitive of order λ of f as the convolution (in generalized sense)

$$f(t)=f(t)*\frac{t^{\lambda-1}}{\Gamma(\lambda)} \quad (\operatorname{Re}\lambda>0) \tag{A.3}$$

where

$$t_{+}^{\lambda-1}\begin{cases}=t^{\lambda-1} & t>0\\ =0 & t\leqq 0\end{cases}$$

and $\Gamma(\lambda)$ is the Gamma function.

It is convenient to put

$$\Phi_{\lambda}=\frac{t^{\lambda-1}}{\Gamma(\lambda)} \tag{A.4}$$

so that

$$f_{\lambda}(t)=f(t)*\Phi_{\lambda} \quad (\operatorname{Re}\lambda>0). \tag{A.5}$$

The validity of this equation may be extended to $\operatorname{Re}\lambda\leqq 0$, provided we intend Φ_{λ} as the generalized function constructed by GEL'FAND and SHILOV (Sect. 3.5.). It results an entire function of λ, such that

$$\Phi_{-n}=\delta^{(n}(t) \quad n=0,1,2,\ldots \tag{A.6}$$

where $\delta(t)$ is denoting the delta function.

As it occurs that

$$f_{-n}(t)=f(t)*\Phi_{-n}=f(t)*\delta^{(n)}(t)=f^{(n)}(t) \tag{A.7}$$

the eq. (A.5) may now be used to generalize the usual notion of derivative of integer order.

One defines the derivative of complex order λ, ($\operatorname{Re}\lambda>0$) of an arbitrary generalized function f concentrated on the half-line $t\geqq 0$ as the convolution (in generalized sense):

$$f_{-\lambda}(t)=f(t)*\Phi_{-\lambda} \quad (\operatorname{Re}\lambda>0) \tag{A.8}$$

and one writes

$$f_{-\lambda}(t)=\frac{d^{\lambda}}{dt^{\lambda}}f(t). \tag{A.9}$$

GEL'FAND and SHILOV point out the fundamental property of the generalized functions Φ_{λ}

$$\Phi_{\lambda}*\Phi_{\mu}=\Phi_{\lambda+\mu} \tag{A.10}$$

or

$$\frac{t_{+}^{\lambda-1}}{\Gamma(\lambda)}*\frac{t_{+}^{\mu-1}}{\Gamma(\mu)}=\frac{t_{+}^{\lambda+\mu-1}}{\Gamma(\lambda+\mu)} \tag{A.10$'$}$$

valid for any complex λ, μ.

From (A.10) some notable implications follow, e.g.

$$\frac{d^\alpha}{dt^\alpha}\left(\frac{d^\beta f}{dt^\beta}\right) = \frac{d^{\alpha+\beta} f}{dt^{\alpha+\beta}} \tag{A.11}$$

$$\frac{d^\lambda}{dt^\lambda}\left(\frac{t_+^\mu}{\Gamma(1+\mu)}\right) = \frac{t_+^{\mu-\lambda}}{\Gamma(1+\mu-\lambda)}. \tag{A.12}$$

Utilising the eq. (A.10), we are going to deduce the explicit expression for the derivative of real order x of a causal ordinary function $f(t)$.

Let be

$$x = n - x', \quad \text{where} \quad n = 1, 2, \ldots \quad 0 < x' < 1 \tag{A.13}$$

then

$$\begin{aligned} \frac{d^x}{dt^x} f(t) &= f(t) * \Phi_{-x} = f(t) * \Phi_{-n+x'} = f(t) * (\Phi_{-n} * \Phi_{x'}) \\ &= (f(t) * \Phi_{-n}) * \Phi_{x'} = f^{(n)}(t) * \Phi_{x'}. \end{aligned}$$

As $x' > 0$, $\Phi_{x'}$ is the ordinary function $t_+^{x'-1}/\Gamma(x')$ and the convolution appearing in the last member is an ordinary one. Then we may write

$$\frac{d^x}{dt^x} f(t) = \frac{1}{\Gamma(x')} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{1-x'}}\, d\tau \quad (0 < x' < 1). \tag{A.14}$$

Putting

$$v = 1 - x' \tag{A.15}$$

we obtain the alternative form, used by CAPUTO [5]:

$$\frac{d^x}{dt^x} f(t) = \frac{1}{\Gamma(1-v)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^v}\, d\tau \quad (0 < v < 1) \tag{A.16}$$

We recall that, for hypothesis, the eqs. (A.14), (A.15) hold in the open intervals $n-1 < x < n$ $(n = 1, 2, \ldots)$.

Now we are going to see if these equations are also valid in the extremes of the intervals.

For $x = n - 1 \Rightarrow v = 0$ $(x' = 1)$ we obtain

$$\frac{d^{n-1}}{dt^{n-1}} f(t) = \int_0^t f^{(n)}(\tau)\, d\tau = f^{(n-1)}(t) - f^{(n-1)}(0^+).$$

For $x = n \Rightarrow v = 1$ $(x' = 0)$ we obtain

$$\frac{d^n}{dt^n} f(t) = f^{(n)}(t) * \delta(t) = f^{(n)}(t)$$

as

$$\Phi_0 = \frac{t_+^{-1}}{\Gamma(0)} = \delta(t).$$

Thus we conclude that in general the eqs. (A.14), (A.16) hold for the intervals $n-1<x\leqq n$, namely for $0\leqq x'<1$ or $0<v\leqq 1$.

Finally, viewing for the applications, we point out the expression for the Laplace transform of the derivative of real order

$$\mathscr{L}\left\{\frac{d^x}{dt^x} f(t)\right\} = p^x\, \mathscr{L}\{f(t)\} - \sum_{0=k}^{n-1} p^{(x-1)-k} f^{(k)}(0^+) \quad (n-1<x\leqq n). \tag{A.17}$$

This property was demonstrated by CAPUTO [5].

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